

# Well-connected residuated lattices and residually finite residuated lattices

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## Abstract

The aim of this paper is to study well-connected residuated lattices and residually finite residuated lattices. It is well known that well-connected residuated lattices is not only a main tool for studying  $\mathcal{RL}_{si}$  but also a subdirect irreducible representation object of residuated lattices. Based on the above facts, we both investigate the above two aspects by using some different methods. Meanwhile, we shall also introduce the residually finite residuated lattices and characterize them from algebraic, logical and topological perspectives, respectively.

*Keywords:* Well-connected residuated lattice; Residually finite residuated lattice; Hopfian residuated lattice; Filter

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## 1. Introduction

Monoidal Logic (ML from now on), introduced by U. Höhle [16], is a logic whose algebraic counterpart is the class of residuated lattices. This logic is built up from four primitive connectives  $\&$ ,  $\rightarrow$ ,  $\wedge$ ,  $\vee$  and the truth constant  $\bar{0}$ . The difference with Basic Fuzzy logic (BL from now on) is due to the fact that in ML  $\wedge$  and  $\vee$  are not definable from the others and thus they need to be introduced as primitive connectives. The commutative residuated lattices were first introduced by M. Ward and R.P. Dilworth [28] as generalizations of ideal lattices of rings. Non-commutative residuated lattices, called sometimes pseudo-residuated lattices, bi-residuated lattices or generalized residuated lattices are algebraic counterpart of substructural logics, that is, logics which lack some of three structural rules, namely contraction, weakening and exchange. Complete studies on residuated lattices were developed by H. Ono [19, 20], T. Kowalski [19, 20], P. Jipsen [17], C. Tsirakis [18, 26] and N. Galatos [12, 13]. Firstly, there has been an algebraic tradition in defining them as well as logical tradition. Substructural logics are non-classical logics that are weaker than classical logic, in the sense that they may lack one or more of the structural rules of contraction, weakening and exchange in their Gentzen-style axiomatization.

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These logics encompass a large number of non-classical logics related to computer science (linear logic[14]), linguistics (Lambek Calculus[15]), philosophy (relevant logics[1]), and many-valued reasoning[16]. The main difference between the two approaches is that the algebraic one renders residuated lattices as structures with only one constant in the type (namely, the unit of the monoid), whereas the logical one makes use of four constants (the top, the bottom, the unit, and a certain zero). Secondly, within the logic tradition, the term residuated lattices has been used to denote an even narrower class of algebras: namely, such that the monoid reduct is commutative and its unit and zero coincide, respectively, with the top element and the bottom element of the lattice. Fortunately, in the logical tradition there also has been an alternative nomenclature. Namely, because of their connection with extensions of what is known in logic as full Lambek calculus, the algebras in question have been called *FL*-algebras. Further, among the extensions of full Lambek calculus there are certain important ones which arise by adding one or more of three so-called structural rules of exchange, weakening, or contraction. A notation devised by H. Ono refers to these logics as  $FL_X$ , with  $X$  being an appropriate subset of  $\{e, w, c\}$ . N. Galatos, P. Jipsen, T. Kowalski and H. Ono (see [12] Sect. 3.4) list some extensive of residuated structures (mostly residuated lattices and *FL*-algebras) that appear in algebra and logic.

According to Birkhoff's theorem, each variety is completely determined by its class of subdirectly irreducible members. It is well known that each subdirectly irreducible Heyting algebra is well-connected[10], in [22], L.L. Maksimova give an algebraic characterization of the disjunction property for superintuitionistic logics, by using well-connected Heyting algebras. Following the work of N. Galatos, P. Jipsen, T. Kowalski and H. Ono [12], in this Section 3 we investigate well-connected residuated lattices.

The residual finiteness from a group theory that occurs in algebraic logic[21], topological algebra[24], and modal logic[8]. A logic is finitely approximable if it is characterized by a class of finite algebras(see [8] for more details). As we known, the congruence lattices of residuated lattices have a nice algebraic properties. Note that the residual finiteness of a algebra is closely related to its congruence lattice. Therefore, in Section 4 we study residually finite residuated lattices.

## 2. Preliminaries

In this section, we summarize some definitions and results about residuated lattices.

**Definition 2.1.** [16] An algebraic structure  $\mathcal{L} = (L, \wedge, \vee, \odot, \rightarrow, 0, 1)$  of type  $(2, 2, 2, 2, 0, 0)$  is called a *residuated lattice* if it satisfies the following conditions:

- (1)  $(L, \wedge, \vee, 0, 1)$  is a bounded lattice;
- (2)  $(L, \odot, 1)$  is a commutative monoid;
- (3)  $x \odot y \leq z$  if and only if  $x \leq y \rightarrow z$ , for all  $x, y, z \in L$ , where  $\leq$  is the partial order of the lattice  $(L, \wedge, \vee, 0, 1)$ .

Throughout this paper we will slightly abuse notation  $L$  the universe of a residuated lattice  $\mathcal{L} = (L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ , when there is no chance to confusion.

For convenience of readers, we provide some basic properties of residuated lattices in the following proposition.

**Proposition 2.2.** [27] *In any residuated lattice  $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ , the following properties hold: for any  $x, y, z \in L$ ,*

- ( $R_1$ )  $1 \rightarrow x = x, x \rightarrow 1 = 1$ ;
- ( $R_2$ )  $x \leq y$  if and only if  $x \rightarrow y = 1$ ;
- ( $R_3$ ) if  $x \leq y$ , then  $y \rightarrow z \leq x \rightarrow z, z \rightarrow x \leq z \rightarrow y$  and  $x \odot z \leq y \odot z$ ;
- ( $R_4$ )  $x \odot (x \rightarrow y) \leq y$ ;
- ( $R_5$ )  $x \odot y \leq x \wedge y, x \leq y \rightarrow x$ ;
- ( $R_6$ )  $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z)$ ;
- ( $R_7$ )  $x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z)$ , hence  $x^m \vee y^n \geq (x \vee y)^{mn}$ .

**Definition 2.3.** [27] Let  $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$  be a residuated lattice. A *filter* is a nonempty set  $F \subseteq L$  such that for each  $x, y \in L$ ,

- (i)  $x, y \in F$  implies  $x \odot y \in F$ ,
- (ii) if  $x \in F$  and  $x \leq y$ , then  $y \in F$ .

Note that in a residuated lattice  $L$ , a filter  $F$  of  $L$  is equivalent to a *deductive system*, that is,  $F$  satisfies the following conditions: (i)  $1 \in F$ , and (ii)  $x, x \rightarrow y \in F$  implies  $y \in F$ . We denote by  $\mathcal{F}(L)$  the set of all filters of a residuated lattice  $L$ . With any filter of  $L$  we can associate a congruence relation  $\theta_F$  on  $L$  by defining

$$(x, y) \in \theta_F \text{ if and only if } x \rightarrow y, y \rightarrow x \in F.$$

For any  $x \in L$ , let  $x/F$  be the equivalence class  $x/\theta_F$ . If we denote by  $L/F$  the quotient set  $L/\theta_F$ , then  $L/F$  becomes a residuated lattice with the operations induced from those of  $L$ . A filter  $F$  of  $L$  is called *prime* if  $x \vee y \in F$  it implies that  $x \in F$  or  $y \in F$  for any  $x, y \in L$ .

We denote the set of prime filters of a residuated lattice  $L$  by  $\mathcal{P}(L)$ .

**Theorem 2.4.** [16] *Let  $L$  be a residuated lattice. For every element  $a \in L$  with  $a \neq 1$  there is a prime filter  $P$  in  $L$  with  $a \notin P$ .*

**Theorem 2.5.** [16] *Let  $L$  be a residuated lattice. Then  $\bigcap \mathcal{P}(L) = \{1\}$ .*

Let  $L$  be a residuated lattice. By  $\mathcal{F}(L)(\text{Con}(L))$ , we mean the set of all filters(congruences) of  $L$ . There is a close correspondence between congruences and filters of residuated lattices. For each congruence  $\theta$  on residuated lattice  $L$ , let  $[1]_\theta = \{x \in L : (1, x) \in \theta\}$ . Then  $[1]_\theta$  is a filter of  $L$ , called the filter *determined* by a congruence  $\theta$ . Conversely, for each filter  $F$ ,  $\theta_F = \{(x, y) \in L \times L : (x \rightarrow y) \odot (y \rightarrow x) \in F\}$  is a congruence, called the congruence *determined* by a filter  $F$ . Moreover the following result holds.

**Theorem 2.6.** [12] For every residuated lattice  $L$ , we have

$$\mathcal{F}(L) \cong \text{Con}(L).$$

**Proposition 2.7.** [29] A residuated lattice  $L$  is a subdirect product of a family  $L_i$  of residuated lattices if and only if there is a family  $\{F_i\}_{i \in I}$  of filters of  $L$  such that

- (i)  $L_i \cong L/F_i$  for each  $i \in I$ ;
- (ii)  $\bigcap_{i \in I} F_i = \{1\}$ .

### 3. Well-connected residuated lattices

A *variety* is a class of algebras which is closed under taking homomorphic images, subalgebras and arbitrary direct products. Since the intersection of a nonempty family of varieties is again a variety, we may consider the *variety generated* by any class  $\mathcal{C}$  of algebras, denoted  $\mathbf{V}(\mathcal{C})$ . One of the most celebrated theorems of Birkhoff says that the classes of algebras defined by identities is a precisely variety. It is well known that the class  $\mathcal{RL}$  of residuated lattices is a variety[3]. Moreover,  $\mathcal{RL}$  is a congruence distributive variety. This follows from the fact that each residuated lattice has a lattice reduct. Thus we can show the congruence distributivity for  $\mathcal{RL}$  by the same majority term as for lattices(see[7] Theorem 12.3).

An algebra  $\mathbf{A}$  is (congruence) *permutable* if for all  $\theta, \varphi \in \text{Con}(A)$ ,  $\theta \circ \varphi = \varphi \circ \theta$ . A variety is permutable if every member is permutable.

**Proposition 3.1.** [3] The variety  $\mathcal{RL}$  is permutable.

Recall that an algebra is *arithmetical* iff it is both congruence distributive and permutable. A variety is arithmetical if its every member is arithmetical, and a variety has the *congruence extension property* or *CEP* if for every algebra  $A$  in the variety, for any subalgebra  $B$  of  $A$  and for any congruence  $\theta$  on  $B$ , there exists a congruence  $\theta'$  on  $A$ , such that  $\theta' \cap B^2 = \theta$ .

**Proposition 3.2.** [19] The variety  $\mathcal{RL}$  is arithmetical and has CEP.

Recall that an element  $a$  of a lattice  $L$  is said to be *join-irreducible* if  $x \vee y = a$  implies that  $x = a$  or  $y = a$ . Dually, an element  $a$  of  $L$  is call *meet-irreducible* if  $a = x \wedge y$  implies that  $a = x$  or  $a = y$ . The element  $a$  is *completely meet-irreducible* is  $a \neq 1$  and whenever  $a = \bigwedge_{i \in I} b_i$ , there is a  $j \in I$  such that  $a = b_j$ . We shall say that the join  $\bigvee_{i=1}^m x_i$  is *irredundant* if it is such that for every  $k$ ,

$$\bigvee_{i=1}^m x_i > x_1 \vee \cdots \vee x_{k-1} \vee x_{k+1} \vee \cdots \vee x_m = \bigvee_{j \neq k}^m x_j.$$

Roughly speaking, a join is irredundant if the removal of any term results in something smaller. If

$$\bigvee_{i=1}^m x_i = \bigvee_{i \neq k}^m x_i$$

then we say that the element  $x_k$  is redundant.

For distributive lattices with the descending chain condition we have the following strengthening of the Kurosh-Ore Theorem.

**Theorem 3.3.** [6] *If  $L$  is a distributive lattice that satisfies the descending chain condition, then every element of  $L \setminus \{0\}$  can be expressed uniquely as an irredundant join of join-irreducibles.*

**Definition 3.4.** A filter  $F$  of a residuated lattice  $L$  is said to be *join-irreducible* if  $F$  is a join-irreducible element of the lattice  $(\mathcal{F}(L), \subseteq)$ , dually,  $F$  is called *meet-irreducible* if it is a meet-irreducible element of the lattice  $(\mathcal{F}(L), \subseteq)$ .

The following result is a rather direct application of the more general result (see [25] Theorem 1.9). We include the complete proof for the readers convenience.

**Theorem 3.5.** *Let  $L$  be a residuated lattice. Then the meet-irreducible filters of  $L$  are precisely prime filters.*

**Proof.** Let  $P$  be a meet-irreducible filter of  $L$  and  $x \vee y \in P$ . If  $x \notin P$  and  $y \notin P$ , then we claim that  $P = (x \vee P) \cap (y \vee P)$ , where  $x \vee P = \langle x \rangle \vee P = \{z : x^n \odot a \leq z \text{ for } a \in P \text{ \& } n \in \mathbb{Z}^+\}$ . It suffices to show that  $(x \vee P) \cap (y \vee P) \subseteq P$ , since the other inclusion is obvious. Let  $z \in (x \vee P) \cap (y \vee P)$ . Thus there exist  $a \in P$  and  $n, m \in \mathbb{Z}^+$  such that  $x^n \odot a \leq z$  and  $y^m \odot a \leq z$ , by Proposition 2.2 ( $R_7$ ), it follows that  $(x \vee y)^{nm} \odot a \leq z$ . Therefore, we have  $z \in P$ , which contradicts the meet-irreducibility of  $P$ . Conversely, suppose that  $P$  is a prime filter of  $L$ . Since  $(\mathcal{F}(L), \subseteq)$  is a distributive lattice, it suffices to show that  $P$  is a meet-prime element of  $(\mathcal{F}(L), \subseteq)$ . If  $F \cap G \subseteq P$  with  $F \not\subseteq P$  and  $G \not\subseteq P$ . Then there are  $x \in F$  and  $y \in G$  such that  $x \notin P, y \notin P$ . Thus, by the primality of  $P$ , we have  $x \vee y \in F \cap G$  and  $x \vee y \notin P$ . This shows that  $F \cap G \not\subseteq P$ , which is a contradiction. Therefore,  $F \subseteq P$  or  $G \subseteq P$ , which completes the proof.

In the following we conclude that a property of prime filters of residuated lattices which is similar to those of commutative rings[2].

**Corollary 3.6.** *Let  $F_1, \dots, F_n$  be filters of a residuated lattice  $L$  and  $P$  be a prime filter containing  $\bigcap_{i=1}^n F_i$ . Then  $P \supseteq F_i$  for some  $i$ . If  $P = \bigcap_{i=1}^n F_i$ , then  $P = F_i$  for some  $i$ .*

In what follows, we give another way to proof Theorem 2.5.

**Corollary 3.7.** *Let  $L$  be a residuated lattice. Then  $\bigcap \mathcal{P}(L) = \{1\}$ .*

**Proof.** Let denote  $\mathcal{CMF}(L)$  the set of all completely meet-irreducible filters of  $L$ . Clearly,  $\{1\} = \bigcap \mathcal{CMF}(L)$  since  $(\mathcal{F}(L), \subseteq)$  is an algebraic lattice. By Theorem 3.5,  $\{1\} = \bigcap \mathcal{P}(L)$  since every completely meet-irreducible filter is a meet-irreducible filter and every filter is the meet of a set of completely meet-irreducible filters([23] Theorem 2.19).

If a residuated lattice  $L$  with the descending chain condition ( $DCC$ ) we have the following theorem which is similar to Theorem 3.3.

**Theorem 3.8.** *If  $L$  is a residuated lattice that satisfies the descending chain condition then every element of  $\mathcal{F}(L) \setminus \{1\}$  can be expressed uniquely as an irredundant join of join-irreducible filters.*

**Proof.** It directly follows from Proposition 3.2 and Theorem 3.3.

**Proposition 3.9.** *[12] A residuated lattice  $L$  is subdirectly irreducible if and only if it satisfies a condition: there is an  $a < 1$  such that, for all  $x < 1$ ,  $x^n \leq a$  for some positive  $n$ .*

**Proposition 3.10.** *Let  $L$  be a residuated lattice. If  $L$  has an element  $a \neq 1$  and  $x \leq a$  for each  $1 \neq x \in L$  and  $a$  is not a nilpotent element (i.e.,  $L \setminus \{1\}$  has a maximum not nilpotent element), then  $L$  is subdirectly irreducible.*

**Proof.** Let  $a \in L$  satisfy the indicate condition. Then the filter  $F = [a]$  is obviously the smallest of all filters distinct from the unit filter  $[1]$ . By Theorem 2.7, there is the smallest nontrivial congruence of  $L$ . Therefore, by ([7] Theorem 8.4),  $L$  is subdirectly irreducible.

**Proposition 3.11.** *Let  $L$  be a non-trivial residuated lattice. If  $L$  is subdirectly irreducible, then the smallest not unit filter is a principal filter.*

**Proof.** Let  $L$  be subdirectly irreducible. By ([7] Theorem 8.4) and Theorem 2.7 there exists a filter  $F$  which is the smallest of all filters distinct from  $[1]$ . The filter  $F$  is principal, i.e.,  $F = [a]$  for some  $a \in L$ . Indeed, for any  $1 \neq a \in F$ , it follows that  $[a] \subseteq F$  and then  $[a] = F$ .

**Corollary 3.12.** *Let  $L$  be a non-trivial residuated lattice.  $L$  is subdirectly irreducible if and only if there is a smallest not unit principal filter in  $L$ .*

**Proof.** The “only if” part follows from above proposition. Now suppose that the condition is true, i.e.,  $[a]$  is the smallest not unit principal filter of  $L$ , any  $x \neq 1$  belong to  $L$  and  $x$  is not a nilpotent element, we have  $[a] \subseteq [x]$ . This has prove the condition of Proposition 3.10 holds, which completes the proof.

In what follows, we can conclude that  $1$  is a join-irreducible element if and only if unit filter  $\{1\}$  is a prime filter.

**Proposition 3.13.** *Let  $L$  be a residuated lattice. Then  $1$  is a join-irreducible element if and only if  $\{1\}$  is a prime filter.*

**Proof.** The proof is straightforward.

**Proposition 3.14.** *Let  $L$  be a residuated lattice. If  $P$  is a prime filter of  $L$ , then the top element  $1/P = P$  of the quotient residuated lattice  $L/P$  is a join-irreducible element.*

**Proof.** Suppose that  $x/P \vee y/P = 1/P$ . Then we have that  $x/P \vee y/P = x \vee y/P = 1/P$ . Thus since  $1 \rightarrow (x \vee y) \in P$  and  $1 \in P$ , it follows that  $x \vee y \in P$ . Note that  $P$  is a prime filter, so we have  $x \in P$  or  $y \in P$ . Therefore, we get  $x/P = 1/P$  or  $y/P = 1/P$ .

**Proposition 3.15.** *Let  $P$  be a prime filter of a residuated lattice  $L$ . If  $L$  is finite, then quotient residuated lattice  $L/P$  is subdirectly irreducible.*

**Proof.** By hypothesis and Theorem 3.5, the prime filter  $P$  is completely meet-irreducible in  $(\mathcal{F}(L), \subseteq)$ . Thus, by ([4] Theorem 3.23.), it follows that  $L/P$  is subdirectly irreducible.

**Definition 3.16.** [12] A residuated lattice  $L$  is said to *well-connected* if 1 of  $L$  is a join-irreducible element. The class of all well-connected residuated lattices will be denote by  $\mathbf{WcRL}$ .

Note that each linear residuated lattice is well-connected and, by Proposition 3.14, the quotient residuated lattice associated to prime filter is a well-connected residuated lattice. Also, if a well-connected residuated lattice satisfies prelinearity (i.e.,  $(x \rightarrow y) \vee (y \rightarrow x) = 1$ ), then it is linear. Therefore, well-connectedness is a generalization of linearity. Recall that a *MTL-algebra* is a residuated lattice satisfies prelinearity. Thus a *MTL-algebra* is linear if and only if it is well-connected.

**Theorem 3.17.** *Up to isomorphism, well-connected residuated lattices are precisely the quotient residuated lattice via the prime filters.*

**Proof.** By the above argument, we have  $L/P$  is a well-connected residuated lattice for a prime filter  $P$  of  $L$ . Conversely, suppose that  $L$  is a well-connected residuated lattice. By Proposition 3.13 and the definition of well-connected, we get  $\{1\}$  is a prime filter. Since  $L \cong L/\{1\}$ , whence we complete the proof.

Recall that  $h : A \rightarrow B$  is a homomorphism of two residuated lattices  $A, B$ , we shall use the symbol  $Ker(h)$  to denote  $Ker(h) = \{x \in A \mid h(x) = 1\}$  the kernel of  $h$ . In the next lemma we summarize, for further reference, some easy relations between ideals and kernels of homomorphisms.

**Lemma 3.18.** *Let  $A, B$  be residuated lattices, and  $h : A \rightarrow B$  a homomorphism. Then the following properties hold:*

- (1) *for each filter  $F$  of  $B$ , the set  $h^{-1}(F) = \{x \in A \mid h(x) \in F\}$  is a filter of  $A$ , thus in particular,  $Ker(h) \in \mathcal{F}(A)$ ;*
- (2)  *$h(x) \leq h(y)$  if and only if  $x \rightarrow y \in Ker(h)$ ;*
- (3)  *$h$  is injective if and only if  $Ker(h) = \{1\}$ ;*
- (4)  *$Ker(h) \neq A$  if and only if  $B$  is nontrivial;*



- (5)  $\text{Ker}(h)$  is a prime filter if and only if  $B$  is nontrivial and the image  $h(A)$ , as a subalgebra of  $B$ , is a well-connected residuated lattice.

**Proof.** The proof is straightforward.

**Proposition 3.19.** *If  $L$  is a well-connected MTL-algebra, then all proper filters of  $L$  are prime filters.*

**Proof.** Suppose that  $F$  is a proper filter of  $L$ . Then  $L$  is a linear residuated lattice, and thus we have  $L/F$  is also a linear residuated lattice. Since  $h_F : L \rightarrow L/F (x \mapsto x/F)$  is a surjective homomorphism and a linear residuated lattice is well-connected, and hence, by Lemma 3.18,  $\text{Ker}(h_F) = F$  must be a prime filter.

As an immediate consequence we also have all proper filters of a linear residuated lattice are prime filters.

**Proposition 3.20.** *Let  $L$  be a finite residuated lattice. If  $L$  is well-connected, then it is subdirectly irreducible.*

**Proof.** Let  $L$  be well-connected residuated lattice. Note that  $L/\{1\} \cong L$ , thus by Proposition 3.15, we have  $L$  is subdirectly irreducible.

In the following we give another way to prove the well known result in [12].

**Theorem 3.21.** *Let  $L$  be a residuated lattice. If  $L$  is subdirectly irreducible, then it is well-connected.*

**Proof.** By Theorem 2.5, we have  $\bigcap \mathcal{P}(L) = \{1\}$ . Since  $L$  is subdirectly irreducible, that is,  $\{1\}$  is completely meet-irreducible, it follows that  $\{1\} \in \mathcal{P}(L)$ , therefore,  $\{1\}$  is a prime filter. Thus by the definition of well-connected residuated lattices, we have  $L$  is a well-connected residuated lattice.

**Corollary 3.22.**  $\mathcal{RL}_{si} \subseteq \mathbf{WcRL}$ .

**Corollary 3.23.** *Let  $L$  be a finite residuated lattice. Then  $L$  is well-connected if and only if it is subdirectly irreducible.*

**Proof.** It immediately follows from Proposition 3.20 and Theorem 3.21.

**Remark 3.24.** *It is easily seen that every linear residuated lattice is well-connected, hence, by Corollary 3.23, every finite linear residuated lattice is subdirectly irreducible. Thus we have given a complete characterization of finite subdirectly irreducible members of the variety  $\mathcal{RL}$ .*

Let  $\mathcal{JF}(L)$  denote the set of join-irreducible filters of a residuated lattice  $L$ .

**Proposition 3.25.** *If  $L$  is a well-connected residuated lattice that satisfies the descending chain condition, then  $(\mathcal{JF}(L), \supseteq)$  is cofinal in  $(\mathcal{F}(L), \supseteq)$ .*



**Proof.** It immediately follows from ([29] Lemma 5.14.).

In what follows, we give a result that is similar to Chang's Subdirect Representation Theorem[9] in order to prove that an equation holds in all residuated lattices it is sufficient to check it holds in all well-connected residuated lattices.

**Theorem 3.26.** (*Subdirect Representation Theorem*) *Every nontrivial residuated lattice is a subdirect product of well-connected residuated lattices.*

**Proof.** Let  $L$  be a residuated lattice. By Theorem 2.5, we have that  $\bigcap \mathcal{P}(L) = \{1\}$ . Notice that the facts of Proposition 2.7 and each prime filter  $P$  of  $L$  the quotient residuated lattice  $L/P$  is well-connected, this is precisely assertion of the theorem.

In view of the above fact we can get the following result, which is important in studying the variety  $\mathcal{RL}$ . Note that the class  $\mathbf{WcRL}$  is not a variety, since it is not closed under the direct product of well-connected residuated lattices.

**Corollary 3.27.**  $\mathcal{RL} = \mathbf{V}(\mathbf{WcRL})$ .

**Lemma 3.28.** [16] *Every proper filters of a residuated lattice is an intersection of prime filters.*

Given an algebra  $A$  and a nonempty family  $(\theta_i)_{i \in I}$  of congruences on  $A$ , there is a natural injective homomorphism  $A / \bigcap_{i \in I} \theta_i \rightarrow \prod_{i \in I} A / \theta_i$ .

**Proposition 3.29.** *Every nontrivial quotient residuated lattice can be embedding the direct product of well-connected residuated lattices.*

**Proof.** It follows from Lemma 3.28 and the above fact.

It is well known that in a  $MTL$ -algebra  $L$ , if  $P$  is a prime filter of  $L$ , then  $L/P$  is a linear  $MTL$ -algebra. Thus, by the above Subdirect Representation Theorem, we deduce that the following well known result about completeness of  $MTL$  with respect to linearly ordered  $MTL$ -algebras given by F. Esteva and L. Godo[11].

**Corollary 3.30.** *Every nontrivial  $MTL$ -algebra can be embedding the direct product of linear  $MTL$ -algebras.*

Let  $\{A_i : i \in I\}$  be an indexed family of algebras of the same type. By an *ultrafilter* on  $I$  we mean an ultrafilter of  $\mathcal{P}(I)$ , viewed as Boolean algebra. Let  $\mathcal{U}$  be an ultrafilter on  $I$ . For elements  $a$  and  $b$  of the direct product  $A = \prod_{i \in I} A_i$ , we define their *equalizer*,  $\llbracket a = b \rrbracket = \{i \in I : a(i) = b(i)\}$ . We define

$$\eta_{\mathcal{U}} = \{(a, b) \in A^2 : \llbracket a = b \rrbracket \in \mathcal{U}\}.$$

It is straightforward to verify that  $\eta_{\mathcal{U}}$  is a congruence relation on  $A$ .

**Theorem 3.31.** *Let  $L = \varprojlim_{i \in I} L_i$  be a profinite residuated lattice. Then for every prime filter  $P$  on  $L$ , there is a filter  $F$  of  $\prod_{i \in I} L_i$  such that  $F \upharpoonright L \subseteq P$ .*

**Proof.** According to Theorem 3.5,  $P$  is meet-irreducible in lattice  $(\mathcal{F}(L), \subseteq)$ , by Theorem 2.7, we have  $\theta_P$  is meet-irreducible in the lattice  $(Con(L), \subseteq)$ . Since  $(Con(L), \subseteq)$  is congruence-distributive, thus this satisfies the conditions of Jónsson's Lemma (e.g., see [4] Lemma 5.9), hence there is an ultrafilter  $\mathcal{U}$  over  $I$  such that  $\eta_{\mathcal{U}} \upharpoonright L \subseteq \theta_P$ . Let  $F = [1]_{\eta_{\mathcal{U}}}$ . Thus, we have  $[1]_{\eta_{\mathcal{U}}} = F \upharpoonright L \subseteq P$ , since  $[1]_{\theta_P} = P$ .

Let  $\mathcal{K}$  be a class of algebras. Let  $\mathbf{P}_{\mathbf{u}}(\mathcal{K})$  denote the class of all algebras isomorphic to ultraproducts of members of  $\mathcal{K}$ . Also, we denote by  $\mathcal{K}_{si}$  the class of all subdirectly irreducible members of  $\mathcal{K}$ .

**Theorem 3.32.**  $\mathcal{RL}_{si} \subseteq \mathbf{HSP}_{\mathbf{u}}(\mathbf{WcRL})$ .

**Proof.** It follows from Corollary 3.27 and ([4] Theorem 5.10).

#### 4. Residually finite residuated lattices

This section is devoted to the study of residually finite residuated lattices. As their name suggests it, residually finite residuated lattices generalize finite residuated lattices. They are defined as being the residuated lattices whose elements can be distinguished after taking finite quotients.

**Definition 4.1.** A residuated lattice  $L$  is called *residually finite* if for each element  $g \in L$  with  $g \neq 1_L$ , there exist a finite residuated lattice  $F$  and a homomorphism  $\phi : L \rightarrow F$  such that  $\phi(g) \neq 1_F$ .

**Proposition 4.2.** *Let  $L$  be a residuated lattice. Then the following conditions are equivalent:*

- (a)  $L$  is residually finite;
- (b) for all  $g, h \in L$  with  $g \neq h$ , there exist a finite residuated lattice  $F$  and a homomorphism  $\phi : L \rightarrow F$  such that  $\phi(g) \neq \phi(h)$ .

**Proof.** The fact that (b) implies (a) is obvious, since (b) gives (a) by taking  $h = 1_L$ . Conversely, suppose that  $L$  is residually finite. Let  $g, h \in L$  with  $g \neq h$ . Then  $g \rightarrow h \neq 1$  or  $h \rightarrow g \neq 1$ . Without loss of generality we can assume that  $g \rightarrow h \neq 1$ , then there exist a finite residuated lattice  $F$  and a homomorphism  $\phi : L \rightarrow F$  such that  $\phi(g \rightarrow h) \neq 1$ . Since  $\phi(g \rightarrow h) = \phi(g) \rightarrow \phi(h) \neq 1$ , it follows that  $\phi(g) \neq \phi(h)$ . Therefore, (a) implies (b).

**Example 4.3.** *Every finite residuated lattice is residually finite.*

Given a residuated lattice  $L$ , the intersection of all filters of finite index of  $L$  is called the *residual filter* (or *profinite kernel*) of  $L$ .

**Proposition 4.4.** *Let  $L$  be a residuated lattice and let  $N$  denote the residual filter of  $L$ . Then  $L$  is residually finite if and only if  $N = \{1\}$ .*

**Proof.** Let  $L$  be residually finite. For  $a \in L$  with  $a \neq 1$ , there exist finite residuated lattice  $F_a$  and a homomorphism  $\phi_a : L \rightarrow F_a$  such that  $\phi_a(a) \neq 1$ . Thus, by the fundamental theorem of homomorphism of residuated lattices, we have  $L/\text{Ker}(\phi_a) \cong F_a$ , hence  $\text{Ker}(\phi_a)$  is a finite index filter for any  $a \in L \setminus \{1\}$ . It follows that

$$N \subseteq \bigcap_{a \in L \setminus \{1\}} \text{Ker}(\phi_a) = \{1\}.$$

The converse is obvious.

In the following we study the stability properties of residually finite residuated lattices.

**Proposition 4.5.** *Every subalgebra of a residually finite residuated lattice is residually finite.*

**Proof.** Let  $L$  be a residually finite residuated lattice and let  $H$  be a subalgebra of  $L$ . Let  $h \in H$  such that  $h \neq 1$ . Since  $L$  is residually finite, there exist a finite residuated lattice  $F$  and a homomorphism  $\phi : L \rightarrow F$  such that  $\phi(h) \neq 1_F$ . If  $\phi' : H \rightarrow F$  is the restriction of  $\phi$  to  $H$ , we have  $\phi'(h) = \phi(h) \neq 1_F$ . Consequently,  $H$  is residually finite.

**Proposition 4.6.** *Let  $(L_i)_{i \in I}$  be a family of residually finite residuated lattices. Then their direct product  $L = \prod_{i \in I} L_i$  is residually finite.*

**Proof.** Let  $g = (g_i)_{i \in I} \in L$  such that  $g \neq 1_L$ . Then there exists  $i_0 \in I$  such that  $g_{i_0} \neq 1_{L_{i_0}}$ . Since  $L_{i_0}$  is residually finite, we can find a finite residuated lattice  $F$  and a homomorphism  $\phi : L_{i_0} \rightarrow F$  such that  $\phi(g_{i_0}) \neq 1_F$ . Consider the homomorphism  $\phi' : L \rightarrow F$  defined by  $\phi' = \pi \circ \phi$ , where  $\pi : L \rightarrow L_{i_0}$  is the projection onto  $L_{i_0}$ . We have  $\phi'(g) = \phi(g_{i_0}) \neq 1_F$ . Consequently,  $L$  is residually finite.

**Proposition 4.7.** *The homomorphic image of a residually finite residuated lattice is residually finite.*

**Proof.** It follows from the fundamental theorem of homomorphism of residuated lattices.

**Theorem 4.8.** *The class of residually finite residuated lattices is a variety, and denoted it by  $\mathbf{RfRL}$ .*

**Proof.** It follows from Propositions 4.5, 4.6 and 4.7.

**Corollary 4.9.** *Let  $L$  be a residuated lattice. Then the following conditions are equivalent:*

- (a)  $L$  is residually finite;
- (b)  $L$  is isomorphic to a subalgebra of the direct product  $\prod_{i \in I} L_i$  of a family  $(L_i)_{i \in I}$  of finite residuated lattices.

**Proof.** The fact that (b) implies (a) follows from Proposition 4.3, Proposition 4.5 and Proposition 4.6. Conversely, suppose that  $L$  is residually finite. Then, for each  $g \in L \setminus \{1_L\}$ , we can find a finite residuated lattice  $F_g$  and a homomorphism  $\phi_g : L \rightarrow F_g$  such that  $\phi_g(g) \neq 1_{F_g}$ . Consider the residuated lattice

$$H = \prod_{g \in L \setminus \{1_L\}} F_g.$$

The homomorphism  $\psi : L \rightarrow H$  defined by

$$\psi = \prod_{g \in L \setminus \{1_L\}} \phi_g$$

is injective. Therefore,  $L$  is isomorphic to subalgebra of  $H$ . This shows that (a) implies (b).

In the following we deduce that the class of residually finite residuated lattices is closed under taking inverse limits.

**Proposition 4.10.** *If a residuated lattice  $L$  is the limit of a inverse system of residually finite residuated lattices, then  $L$  is residually finite.*

**Proof.** Let  $\{L_i, \varphi_{ij}, I\}$  be a inverse system of residually finite residuated lattices such that  $L = \varprojlim_{i \in I} L_i$ . By construction of inverse limit,  $L$  is a subalgebra of the residuated lattice  $\prod_{i \in I} L_i$ . we deduce that  $L$  is residually finite by using Proposition 4.5 and Proposition 4.6.

Recall that a residuated lattice  $L$  is called *profinite* if it is isomorphic to the inverse limit of an inverse system of filter residuated lattices. The class of this algebras denoted by **Pro** $\mathcal{RL}$ .

An immediate consequence of Proposition 4.10 is the following results:

**Corollary 4.11.** *Every profinite residuated lattice is residually finite.*

**Corollary 4.12.** **Pro** $\mathcal{RL} \subseteq \mathbf{Rf}\mathcal{RL}$

Recall that an algebra  $A$  is *finitely approximable* if  $A$  is isomorphic to a subalgebra of a product of finite algebras [21]. It follows that  $A$  is finitely approximable iff  $A$  is a subdirect product of its finite homomorphic images.

**Proposition 4.13.** *A residuated lattice is residually finite if and only if it is finitely approximable.*

**Proof.** It immediately follows from Corollary 4.9.

We denote the class of finite residuated lattices by **f** $\mathcal{RL}$ . Thus, we have the following result:

**Theorem 4.14.** **Rf** $\mathcal{RL} = \mathbf{V}(\mathbf{f}\mathcal{RL})$ .

**Remark 4.15.** In view of the above characterization of  $\mathbf{RfRL}$ , every  $L \in \mathbf{RfRL}$  there exists a family  $(L_i)_{i \in I}$  of finite residuated lattices such that  $L$  can embed  $\prod_{i \in I} L_i$ , by Theorem 3.31, it follows that if  $P$  is a prime filter of  $L$ , then there is a filter  $F$  of  $\prod_{i \in I} L_i$  such that  $F \upharpoonright L \subseteq P$ . Also, we deduce that  $\mathbf{RfRL}_{si} \subseteq \mathbf{HSP}_u(\mathbf{fRL})$ .

**Proposition 4.16.** Let  $L$  be a residuated lattice. The canonical map  $e_L : L \rightarrow \widehat{L}$  is monohomomorphic iff  $L$  is finitely approximable.

**Proof.** If  $e_L$  is injective, then it follows from the definition of  $\widehat{L}$  that  $L$  is isomorphic to a subalgebra of a product of finite algebras, thus  $L$  is finitely approximable. Conversely, if  $L$  is finitely approximable, then  $L$  is a subdirect product of the collection  $\{L/\theta : \theta \in I\}$  of finite homomorphic images of  $L$ . Therefore,  $a \neq b$  in  $L$  implies that there exists  $\theta \in I$  such that  $[a]_\theta \neq [b]_\theta$ . Thus, the images of  $a$  and  $b$  in the inverse limit of  $\{L/\theta, \varphi_{\theta\varphi}, I\}$  are different, and so  $e_L$  is injective.

**Proposition 4.17.** Let  $L$  be a residuated lattice and  $\widehat{L}$  be the profinite completion of  $L$ . Then  $\widehat{L}$  is complete.

**Proof.** To show  $\widehat{L}$  is complete, suppose  $X \subseteq \widehat{L}$ , and denote each  $a \in X$  by  $a = (a_\theta)_{\theta \in I}$ . For each  $\theta \in I$  we set  $b_\theta = \bigvee \pi_\theta(X)$ , and then set  $b = (b_\theta)_{\theta \in I}$ . It is then clear that  $b = \bigvee X$  in  $\prod_{\theta \in I} L/\theta$ . It suffices to show  $b \in \widehat{L}$ . To see this, if  $\psi \subseteq \chi$ , then  $\varphi_{\psi\chi}(a_\psi) = a_\chi$  for each  $a \in X$ . Therefore, using that the quotients  $L/F$  are finite,

$$\varphi_{\psi\chi}(b_\psi) = \varphi_{\psi\chi}\left(\bigvee_{L/\psi} \pi_\psi(X)\right) = \bigvee_{L/\chi} \varphi_{\psi\chi}\pi_\psi(X) = \bigvee_{L/\chi} \pi_\chi(X) = b_\chi,$$

and so  $b \in \widehat{L}$ .

**Remark 4.18.** Suppose that an algebra  $A$  has a lattice reduct. Then  $\widehat{A}$  is complete. Likewise, if every member of a variety has lattice reduct, then its profinite algebras always complete.

In what follows, we get some results which are similar to the notion of residually finite algebras.

**Lemma 4.19.** Let  $L$  be a residuated lattice and  $a \in L$  with  $a \neq 1$ . Then there exists a well-connected residuated lattice  $H$  and a surjective homomorphism  $f : L \rightarrow H$  such that  $f(a) \neq f(1)$ .

**Proof.** Let  $a$  be an element of  $L$  and  $a \neq 1$ . By Theorem 2.4, there is a prime filter  $P$  of  $L$  such that  $a \notin P$ . Suppose that  $f : L \rightarrow L/P$  is a canonical homomorphism. Obviously,  $f$  is surjective. According to Proposition 3.14,  $L/P$  is a well-connected residuated lattice, hence it suffices to show  $f(a) \neq f(1)$ . Note that  $f(a) = a/P$  and  $f(1) = 1/P$  and  $a \notin P$ , it follows that  $a/P \neq 1/P$ .

**Theorem 4.20.** *Let  $L$  be a residuated lattice and  $a, b \in L$  with  $a \neq b$ . Then there exists a well-connected residuated lattice  $H$  and a surjective homomorphism  $f : L \rightarrow H$  such that  $f(a) \neq f(b)$ .*

**Proof.** Note that  $a \neq b$  if and only if  $a \rightarrow b \neq 1$  or  $b \rightarrow a \neq 1$ . Without loss of generality we can assume  $a \rightarrow b \neq 1$ . Thus, by the above lemma, there is a prime filter  $P$  of  $L$  such that  $a \rightarrow b \notin P$ , whence we have  $a/P \rightarrow b/P \neq 1/P$ . Therefore, we obtain that  $f(a) = a/P \neq b/P = f(b)$ .

**Definition 4.21.** A residuated lattice  $L$  is called *Hopfian* if every surjective endomorphism of  $L$  is injective.

**Example 4.22.** (a) *Every finite residuated lattice is Hopfian.*

(b) *Every simple residuated lattice is Hopfian.*

**Lemma 4.23.** *Let  $L$  be a finitely generated residuated lattice and let  $H$  be a finite residuated lattice. Then the set  $\text{Hom}(G, H)$  is finite.*

**Proof.** The proof is straightforward.

**Theorem 4.24.** *Every finitely generated residually finite residuated lattice is Hopfian.*

**Proof.** Let  $L$  be a finitely generated residually finite residuated lattice. Suppose that  $\psi : L \rightarrow L$  is a surjective endomorphism. Let  $F$  be a filter of finite index of  $L$  and let  $\rho : L \rightarrow L/F$  denote the canonical homomorphism. Consider the map

$$\Phi : \text{Hom}(L, L/F) \rightarrow \text{Hom}(L, L/F)$$

defined by  $\Phi(u) = u \circ \psi$  for all  $u \in \text{Hom}(L, L/F)$ . The map  $\Phi$  is injective since  $\psi$  is surjective by our hypothesis. As the set  $\text{Hom}(L, L/F)$  is finite by Lemma 4.23, we deduce that  $\Phi$  is also surjective. In particular, there exists a homomorphism  $u_0 \in \text{Hom}(L, L/F)$  such that  $\rho = u_0 \circ \psi$ . This implies  $\text{Ker}(\psi) \subseteq \text{Ker}(\rho) = F$ . It follows that  $\text{Ker}(\psi)$  is contained in the intersection of all filters of finite index of  $L$ . As  $L$  is residually finite, we deduce that  $\text{Ker}(\psi) = \{1\}$  by Proposition 4.4. Thus  $\psi$  is injective. This shows that  $L$  is Hopfian.

If  $\mathcal{P}$  is a property of residuated lattices, one says that a residuated lattice  $L$  is *virtually  $\mathcal{P}$*  if  $L$  contains a subalgebra of finite index (since the variety  $\mathcal{RL}$  satisfies *CEP*, thus if  $H$  is a subalgebra of  $L$ , then there exists a congruence  $\theta \in \text{Con}(L)$  such that  $H \times H = \theta \upharpoonright H$ , hence we mean  $H$  is a subalgebra of finite index, that is,  $[1]_\theta$  is a finite index filter of  $L$ ) which satisfies  $\mathcal{P}$ .

**Lemma 4.25.** *Let  $H$  be a subalgebra of finite index of a residuated lattice  $L$  and  $F$  be a filter of finite index of  $H$ . Then  $F$  is a filter of finite index of  $L$ .*

**Proof.** Let  $h_1, \dots, h_n$  be a complete set of representatives of the cosets of  $L$  modulo  $H$  and  $k_1, \dots, k_p$  be a complete set of representatives of the cosets of  $H$  modulo  $F$ . Observe that the elements  $h_i k_j, 1 \leq i \leq n, 1 \leq j \leq p$ , form a complete set of representatives of cosets of  $L$  modulo  $F$ . Therefore  $|L/F| < \infty$ .

Compared with Proposition 4.5, in the following we give an opposite approach to find the residually finite residuated lattice.

**Theorem 4.26.** *Every virtually residually finite residuated lattice is residually finite.*

**Proof.** Let  $L$  be a residuated lattice and  $H$  be a subalgebra of finite index of  $L$ . By Lemma 4.25, the intersection of the filters of finite index of  $L$  is contained in the intersection of the filters of finite index of  $H$ . Since a residuated lattice is residually finite if and only if the intersection of its filters of finite index is reduced to the identity element (Proposition 4.4), we deduce that  $L$  is residually finite if  $H$  is residually finite.

**Theorem 4.27.** *Let  $L$  be a infinite simple residuated lattice. Then  $L$  is not residually finite (profinite).*

**Proof.** The only filter of finite index of  $L$  is  $L$  itself. Therefore  $L$  is not residually finite by Proposition 4.4.

**Example 4.28.** *If consider on  $I = [0, 1]$ ,  $\odot$  to be the usual multiplication of real numbers and for  $x, y \in I$ ,*

$$x \rightarrow y = \begin{cases} 1, & x \leq y, \\ y/x, & \text{otherwise.} \end{cases}$$

*Then  $(I, \max, \min, \odot, \rightarrow)$  is a residuated lattice (called Products structure or Gaines structure). Routine calculation shows that it is a infinite simple residuated lattice, thus  $(I, \max, \min, \odot, \rightarrow)$  is neither residually finite nor profinite.*

Recall that a variety is *residually finite* [23] is every subdirectly irreducible member of it is finite. In Example 4.28, since Products structure is a infinite simple residuated lattice, hence, by the definition of variety of residually finite, we have the following result:

**Theorem 4.29.**  $\mathcal{RL}$  is not residually finite.

**Theorem 4.30.**  $\mathbf{RfRL}$  is residually finite.

**Proof.** It follows from Proposition 4.16 and ([5] Proposition 1.5).

**Corollary 4.31.** *Any subvariety of  $\mathbf{RfRL}$  is residually finite.*



Now we summarize the characterizations of residually finite residuated lattices as follows:

**Theorem 4.32.** *Let  $L$  be a residuated lattice. Then the following conditions are equivalent:*

- (a)  $L$  is residually finite;
- (b) the elements of  $L$  can be distinguished after taking finite quotients (Proposition 4.2);
- (c)  $N_L = \{1\}$  (Proposition 4.4);
- (d)  $L$  is finite approximable (Proposition 4.13);
- (e) the canonical map  $e_L : L \rightarrow \widehat{L}$  is injective homomorphic (Proposition 4.16);
- (f) finite index topology of  $L$  is Hausdorff ([29] Theorem 5.2).

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## References

- [1] A.R. Anderson, N.D. Belnap, Entailment: The Logic of Relevance and Necessity, volume 1, Princeton University Press, Princeton, 1975.
- [2] M.F. Atiyah, I.G. MacDonal, Introduction to Commutative Algebra, Addison Wesley, Reading, Mass, 1969.
- [3] R. Belohlavek, Some properties of residuated lattices, Czechoslovak Mathematical Journal, 2003, 53(123):161-171.
- [4] C. Berman, Universal Algebra, Fundamentals and Selected Topics, Pure and Applied Mathematics, vol.301. CRC Press, 2012.
- [5] G. Bezhanishvili, P.J. Morandi, Profinite Heyting algebras and profinite completions of Heyting algebras, Georgian Mathematical Journal, 2009, 16: 29-47.
- [6] T.S. Blyth, Lattice and Ordered Algebraic Structures, Springer-Verlag, London, 2005.
- [7] S. Burris, H. P. Sankappanavar, A Course in Universal Algebra, Springer New York, Graduate Texts in Mathematics, 1981.
- [8] A. Chagrov, M. Zakharyashev, Modal Logic, Oxford Logic Guides 35, Oxford Univ. Press, Oxford, 1997.
- [9] R. Cignoli, I. D'Ottaviano, D. Mundici, Algebraic Foundations of Many-Valued Reasoning, volume 7 of Trends in Logic. Kluwer, Dordrecht, 1999.

- [10] L. Esakia, Heyting Algebras. Duality Theory, Springer, 2019, English translation of the original 1985 book.
- [11] F. Esteva, L. Godo, Monoidal t-norm based logic: towards a logic for left-continuous t-norms, *Fuzzy Sets and Systems*, 2001, 124: 271-288.
- [12] N. Galatos, P. Jipsen, T. Kowalski, H. Ono, Residuated Lattices: An Algebraic Glimpse at Substructural Logic, *Studies in Logic and the Foundations of Mathematics*, Elsevier, 2007.
- [13] N. Galatos, P. Jipsen, Residuated frames with applications to decidability, *Transactions of the American Mathematical Society*, 2013, 365: 1219-1249.
- [14] J.Y. Girard, Linear logic, *Theoretical Computer Science*, 1987, 50: 1-102.
- [15] J. Lambek, The mathematics of sentential structure, *American Mathematical Monthly*, 1958, 65: 154-170.
- [16] U. Höhle, Commutative residuated  $\ell$ -monoids. In: U. Höhle, and E. Klement, (eds), *Non-Classical Logics and their Applications to Fuzzy Subsets*. Kluwer, Dordrecht, 1995.
- [17] P. Jipsen, An overview of generalized basic logic algebras, *Neural Network World*, 2003, 13: 491-500.
- [18] B. Jónsson, C. Tsingajis, Products of classes of residuated structures, *Studia Logica*, 2004, 77: 267-292.
- [19] T. Kowalski, H. Ono, Splitting in the variety of residuated lattices, *Algebra Universalis*, 2000, 44: 283-298.
- [20] T. Kowalski, H. Ono, Residuated Lattices: An algebraic glimpse at logics without contraction, Japan Advanced Institute of Science and Technology, March, 2001.
- [21] A.I. Malcev, *Algebraic Systems*, Springer, New York, 1973.
- [22] L.L. Maksimova, On maximal intermediate logics with the disjunction property, *Studia Logica*, 1986, 45: 69-75.
- [23] R.N. McKenzie, G.F. McNulty, W.F. Taylor, *Algebras, Lattices, Varieties*, Vol.1., Wadsworth, Belmont (Cal.), 1987.
- [24] L. Ribes, P. Zalesskii, *Profinite Groups*. Second edition, *Results in Mathematics Related Areas*, 3rd Series, A Series of Modern Surveys in Mathematics 40, Springer-Verlag, Berlin, 2010
- [25] S.H. Sun, Spectra of monoidal-lattices, *Algebra Universalis*, 1994, 31: 274-292.
- [26] C. Tsingakis, A. Wille, Minimal varieties of involutive residuated lattices, *Studia Logica*, 2006, 83: 407-426.

- [27] E. Turunen, Mathematics Behind Fuzzy Logic, Physica-Verlag, 1999.
- [28] M. Ward, P.R. Dilworth, Residuated lattice, Trans. Am. Math. Soc., 1939, 45: 335-354
- [29] J. Yang, P.F. He, J.T. Wang, On profiniteness and Hausdorffness of topological residuated lattices, 2023, <https://doi.org/10.48550/arXiv.2302.11333>.